On the PRECESSION *of the* EQUINOXES. *By the Rev.* MATTHEW YOUNG, *D. D. S. F. T. C. D. & M. R. I. A.*

IT is univerſally acknowledged, that Sir Iſaac Newton has fallen into ſome error in his calculation of the ſun's force to produce the preceſſion of the equinoxes, making it by one half leſs than the truth: but the particular ſource of this error has not been ſo generally agreed upon. Read April 1, 1797.

THOUGH ſeveral excellent mathematicians, of whom D'Alambert ſeems to have been the firſt, have given genuine ſolutions of this problem, by proceſſes entirely different from each other, perhaps it ſtill may be worth while to endeavour to diſcover diſtinctly in what conſiſts the fallacy of Newton's reaſoning, and whether in ſome of the ſolutions of this curious queſtion, which are received as genuine, there do not lie ſome ſecret and unobſerved errors, which being equal and contrary, compenſate each other, and thus leave the reſult correct, though the premiſes from which it is deduced are faulty.

The first Lemma which Newton premiſes to the inveſtigation of the preceſſion is as follows:

Fig. 1. " If A P E P repreſent the earth, of uniform denſity, de-
" ſcribed with the centre C, poles P, *p*, and equator A E; and
" if with the centre C and radius P C, the ſphere P *a p e* be ſup-
" poſed to be deſcribed; and Q R be a plane perpendicular to
" the right line joining the centres of the ſun and earth; and
" every particle of all the exterior earth P *a p* A P *e*, which is
" higher than the inſcribed ſphere, endeavour to recede on
" either ſide from the plane Q R, and the effort of each particle
" be proportional to its diſtance from the plane; I ſay, firſt,
" that the whole force and efficacy of all the particles in the
" circle of the equator A E, diſpoſed uniformly without the
" ſphere, throughout the whole circumference, in the form of a
" ring, to turn the earth round its centre, is to the whole force
" and efficacy of as many particles placed at the point A of
" the equator which is moſt remote from the plane Q R, to
" move the earth round its centre with a like circular motion,
" as one to two. And that circular motion will be performed
" round an axis lying in the common interſection of the equator
" and the plane Q R."

The demonſtration of this Lemma is given in the Principia, and allowed to be legitimate.

His

His fecond Lemma is as follows:

" The fame things being fuppofed, I fay, fecondly, that the
" whole force and efficacy of all the particles without the fphere
" to turn the earth round its axis, is to the whole force of as
" many particles difpofed uniformly in the form of a ring, in
" the circumference of the circle A E of the equator, to move
" the earth, with a like circular motion, as two to five."

The demonftration of this Lemma is alfo given in the Principles, and is likewife received as unexceptionable.

Lemma 3.

" The fame things being fuppofed, I fay, thirdly, that the
" motion of the earth round the axis already defcribed, com-
" pounded of the motion of all its particles, will be to the
" motion of the aforefaid ring round the fame axis in a ratio,
" which is compounded of the ratio of the quantity of matter
" in the earth to the quantity of matter in the ring, and of the
" ratio of three fquares of the arch of a quadrant of a circle
" to two fquares of the diameter; that is, in a ratio of matter
" to matter, and of the number 925275 to the number
" 1000000."

This Lemma I fhall firft demonftrate in Newton's fenfe, and then correct the conclufion on the principles propofed by Simpfon and Frifi.

By

By the revolution of the circle E A H C, and circumfcribed fquare (fig. 2.) P Q S T round the common axis E H, let there be defcribed a fphere and circumfcribed cylinder. Let the radius A O be $= 1$, the periphery of the circle A E C H $= p$, the ordinate B R $= y$, abfciffa B O $= x$. Then $1 : p :: x : p x$, the periphery of the circle whofe radius is O B; therefore $p x \times 2 y$ will be the furface generated by the ordinate R G, in the revolution of the circle A E C H round the diameter E H: but x will be the meafure of the velocity of the point B, therefore $2 p x^2 y$ will be the momentum of all the particles in that furface; and the fluent of the quantity $2 p x^2 y \dot{x}$ will be the momentum of the entire fphere, when x is equal to the radius A O. But $y = \overline{1 - x^2}|^{\frac{1}{2}}$; therefore the fluxion $x^2 \dot{x} y = x^2 \dot{x} \times \overline{1 - x^2}|^{\frac{1}{2}} = \frac{x^2 \dot{x}}{\overline{1 - x^2}|^{\frac{1}{2}}} - \frac{x^4 \dot{x}}{\overline{1 - x^2}|^{\frac{1}{2}}}$; and the fluent of $\frac{x^2 \dot{x}}{\overline{1 - x^2}|^{\frac{1}{2}}} = \frac{1}{2} \times$ circular arc E R $- \frac{1}{2} x \times \overline{1 - x^2}|^{\frac{1}{2}}$, and the fluent of $-\frac{x^4 \dot{x}}{\overline{1 - x^2}|^{\frac{1}{2}}} = -\frac{3 \times \text{circular arc E R} - \overline{2 x^2 + 3} \times x \times \overline{1 - x^2}|^{\frac{1}{2}}}{8}$; therefore the whole fluent, when $x = 1$, is $\frac{1}{8} \times$ quadrantal arc E A $= \frac{1}{32} p$; and $2 p x^2 \dot{x} \times \overline{1 - x^2}|^{\frac{1}{2}} = \frac{1}{16} p^2$, the motion of the entire fphere.

In a cylinder, the ordinate y becomes $=$ B R $= 1$; therefore the fluxion of the momentum of the cylinder $= 2 p x^2 \dot{x}$, whofe fluent, when $x = 1$, is $\frac{2}{3} p$. Therefore the motion of a cylinder is to the motion

motion of an inſcribed ſphere, revolving round the ſame fixed axis, and with the ſame angular velocity, as $\frac{2}{3}p$ to $\frac{1}{16}p^2$, or as 16 to $\frac{3}{2}p$, that is, as four equal ſquares to three circles inſcribed in them.

LET the quantity of matter in an indefinitely ſlender ring, ſurrounding the ſphere and cylinder at their common contact A O C, be repreſented by the letter m, its velocity will be as A O $= 1$; and its motion $= m$, and therefore the motion of the cylinder is to the motion of the ring as $\frac{2}{3}p$ to m, or as $2p$ to $3m$.

THE motion of the annulus, uniformly continued round the axis of the cylinder, is to its motion revolving uniformly in the ſame periodic time round one of its diameters, as the circumference of a circle to twice the diameter.

FOR (fig. 2) let A R $= z$, and let its fluxion $\dot{z}$ be given, R B $= y$, A B $= x$, and A O $= r$; let the motion be performed round the diameter A C, the velocity of the point R will be as R B or y; therefore the fluxion of the motion of the annulus round the diameter AC, is to the fluxion of the motion round the center O in an immoveable plane, as $\dot{z}y$ to $\dot{z}r$, that is, from the nature of a circle, as $\dot{x}$ to $\dot{z}$; and therefore the motions themſelves are to each other in the ſame ratio, that is, when $x =$ A C, as the diameter to half the circumference, or as twice the diameter to the circumference of a circle.

HENCE, by compounding all theſe ratios, the truth of the Lemma is manifeſt.

BUT

But Simpſon in his miſcellaneous tracts has juſtly obſerved, that though this reaſoning be indiſputably true in Newton's ſenſe, yet there is a difference between the quantity of motion ſo conſidered, and the momentum, whereby a body, revolving round an axis, endeavours to perſevere in its preſent ſtate of motion, in oppoſition to any new force impreſſed, which latter kind of momentum it is that ought to be regarded in computing the alteration of the body's motion in conſequence of ſuch force. In this caſe, every particle is to be conſidered as acting by a lever terminating in the axis of motion; ſo that to have the whole momentum, the moving force of ſuch particle muſt be multiplied into the length of the lever by which it is ſuppoſed to act; whence the momentum of each particle will be proportional to the ſquare of the diſtance from the axis of motion, as it is known to be in finding the center of percuſſion, which depends on the very ſame principles.

The correction ariſing from this change in the proceſs amounts only to about $1\frac{1}{2}''$, as will eaſily appear in the following manner:

The fluxion of the moment of a ſphere, from what has been ſaid already, is $2p\,x^3\,y\,\dot{x}$; from the nature of the circle, $x^2 = 1 - y^2$, as before; therefore $x\dot{x} = -y\dot{y}$, $x^3\dot{x} = y^3\dot{y} - y\dot{y}$, and $2p\,x^3\,y\,\dot{x} = 2p \times \overline{y^4\dot{y} - y^3\dot{y}}$, whoſe fluent is $\frac{4}{15}\,p$, when $y = 1$.

In

In a cylinder, $y = 1$, therefore the fluxion of the moment $= 2\,p\,x^3\,\dot{x}$; whofe fluent is $\frac{1}{2}\,p$, when $x = 1$.

The moment of a ring revolving round its center is double the momentum of the fame ring revolving round one of its diameters. For let $\dot{z}$ be the fluxion of the arch, y the ordinate, and x the abfciffa, radius being unity; $\dot{z}\,y^2$ is the fluxion of the moment of the ring revolving round one of its diameters; but, from the nature of the circle, $\dot{z} = \frac{\dot{x}}{y}$, therefore $\dot{z}\,y^2 = \dot{x}\,y$, which is the fluxion of the area ABR; therefore when $x = 1$, that is, when the arch is equal to $\frac{1}{4}\,p$, the meafure of the moment will be the area of a quadrant; and the meafure of the moment of the entire ring will be equal to the area of the circle, or $\frac{1}{2}\,p$.

If the ring revolve round its center, in an immoveable plane, its moment will be equal to the ring multiplied into the fquare of its radius, that is, equal to p. Therefore the moment in the former cafe is to that in the latter, as $\frac{1}{2}\,p$ to p, or as one to two.

Hence, from what has been demonftrated, the momentum of a fphere is to the momentum of a cylinder, revolving round their axes with the fame angular velocity, as $\frac{4}{15}$ to $\frac{1}{2}$; the momentum of a cylinder is to the momentum of a ring revolving round its centre, in like manner, as $\frac{1}{2}\,p$ to m; and the momentum

of a ring revolving round its centre, is to the momentum of the ſame ring revolving round one of its diameters, as two to one; therefore compounding theſe ratios, and *ex æquo* the momentum of a ſphere revolving round its axis, is to the momentum of a ring revolving round one of its diameters, as $8p$ to $15m$, or as $800000 \times$ quantity of matter in the ſphere, to $1000000 \times$ the quantity of matter in the ring.

If therefore $9'' \, 7''' \, 20^{iv}$, viz. the quantity of the preceſſion, which according to Newton's calculation ariſes from the action of the ſun alone, be encreaſed in the ratio of 925725 to 800000, it will become $10'' \, 33'''$.

But it is well known, that the true quantity of the preceſſion, ariſing from the action of the ſolar force, is nearly double this quantity. Since therefore the correction of this 3d Lemma will not account for the great difference between the reſult of Newton's calculation and the truth, we muſt look for the cauſe of the difference elſewhere. Simpſon is of opinion, that it ariſes from this, that the momentum of a very ſlender ring revolving about one of its diameters, is only the half of what it would be if the revolution were to be performed in a plane, about the centre of the ring; and therefore, that all concluſions, which do not take this into the account, muſt be two little by juſt one half. But it is evident, that this cannot be the true cauſe of the difference, becauſe Newton did actually conſider, that the motion of a ring round one of

of its diameters was lefs than when it revolved round its centre, though he has differed from Simpfon in the ratio which he has affigned of their motions in thefe two cafes; and when the ratio of their motions is admitted to be as one to two, and the other corrections propofed by Simpfon are alfo made, the total error on thefe accounts is found to be but 1, 5″, as has been already fhewn.

MR. MILNER, in his paper on this fubject in the 69th vol. of the Philofophical Tranfactions, agrees with Frifi in thinking, that the error lies in Newton's affumption, that the receffion of the nodes of a rigid annulus and a folitary moon, revolving in the perimeter of the annulus, are equal; whereas in truth, as they affert, (though erroneoufly, as we fhall prefently fhew), the receffion of the latter is but one half of that of the former.

LET us therefore examine particularly whether the receffion of the nodes of a rigid annulus be indeed double the receffion of the nodes of a folitary moon, as has been afferted.

LET A E (Fig. 1.) reprefent the rigid annulus, indefinitely flender, projected into its own diameter, P p its axis; let the line of the nodes be at right angles to S C, the line joining the centres of the fun and earth. From C take the arch C L, and draw L M parallel to D B; let $g =$ the gravity of any given quantity of matter, as a cubic inch; $h =$ the fpace defcribed in 1″ by a

 body

body falling freely by the force of gravity; p = the periphery of a circle whose diameter is unity; also let A C = 1; S. angle D C A = s; Cos. D C A = c; arch C L = z; sine of C L = y. Then L M = cy, and C M = sy.

The disturbing force of the sun is equal to $f \times$ L M (Cor. 17. Prop. 66. Lib. 1. Princip.) and the force of a particle of matter at L to move the annulus about the centre, in the direction P Q A D, is C M $\times f \times$ L M, acting by the power of the lever C M; that is, the force of this quantity of matter at L is = $csfy^2$; therefore the fluxion of the force of the matter in a quadrant of the annulus is $csfy^2\dot{z} = csf \times \frac{y^2\dot{y}}{\sqrt{1-y^2}}$; but the fluent of $\frac{y^2\dot{y}}{\sqrt{1-y^2}}$ is $\frac{1}{2}z - \frac{1}{2}y \times \overline{1-y^2}|^{\frac{1}{2}}$, and therefore the whole fluent is $\frac{1}{2}csfz - \frac{1}{2}csfy \times \overline{1-y^2}|^{\frac{1}{2}}$; and when $y = 1$, the force of the matter in a quadrant of the annulus is $= \frac{csfp}{4}$, and the force of the whole annulus is $pcsf$ = to the simple force $\frac{pcsf}{\sqrt{\frac{1}{2}}}$ acting at the distance $\sqrt{\frac{1}{2}}$ from the centre, that is, at the distance of the centre of gyration from the centre of the annulus. This is the force of the sun, to disturb the annulus, when at the greatest distance from the nodes; call this simple force Fcs.

THE

The quantity of matter in the annulus is $2p$, and the diſtance of the centre of gyration from the centre of the earth is $\sqrt{\frac{1}{2}}$; and by the property of that centre, if the whole matter of the annulus were collected into that point, any force applied to move it about the centre C, would generate the ſame angular velocity, in the ſame time, as it would do in the ring itſelf. And ſince this force Fcs acts at the ſame diſtance $\sqrt{\frac{1}{2}}$ from the centre of the annulus, it is the ſame thing as if it were directly applied to the body to move it. Now to find the motion generated, ſince the ſpace deſcribed in a given time, is as the force directly, and the matter moved inverſely, therefore $g : h :: \frac{pcsf}{2p\sqrt{\frac{1}{2}}} : \frac{hfcs}{2\sqrt{\frac{1}{2}}\,g}$ = the ſpace deſcribed by the centre of gyration in 1″. And $2p\sqrt{\frac{1}{2}}$ (the circumference of the circle whoſe radius is the diſtance of the centre of gyration from the centre of the annulus) : $360^{\circ} ::$ $\frac{hfcs}{2\sqrt{\frac{1}{2}}\,g} : 360 \times \frac{hfcs}{2pg}$ the angle through which the ring is drawn in 1″ by the action of the ſun, when at the greateſt diſtance from the nodes.

But the force of the ſun when at any other diſtance from the nodes, as at H, will be leſs; and the mean quantity of the force may thus be inveſtigated. Draw the great circle pHGP, and making radius = 1, let the arch CH = z, ſine of CH = y; then in the ſphærical triangle CHG, Rad. (1): S. CH (y) :: S. angle DCA (s): S. HG = sy. But it has been already proved, that

that the force of the ſun is equal to F × by the product of the ſine and coſine of his height above the plane of the annulus, therefore the force of the ſun at H is equal to $F\,sy \times \overline{1-s^2 y^2}|^{\frac{1}{2}}$. But this force acts entirely in the plane P G H p, therefore we muſt reſolve it into two forces, one acting in the plane P Q A, which is that we are looking for, the other in the plane P C p, perpendicular to the former; this latter force is deſtroyed by an equal and contrary force, when the ſun is equidiſtant on the other ſide of the line of the nodes; but the other force always acting in the ſame direction, is that only by which the ring is annually affected. The Cos. G H: Cos. angle D C A:: Rad. : Sin. angle H (Cas. 11. Sph. Trig.) and Rad: Sin. angle H:: Sin. C H: Sin. C G (Cas. 2.) ∵ Cos. G H $(\overline{1-s^2 y^2}|^{\frac{1}{2}})$: Cos. D C A (c):: Sin. C H (y): Sin. C G $= \frac{cy}{\overline{1-s^2 y^2}|^{\frac{1}{2}}}$. Then, to find the part of the force acting in the plane P Q A, Rad. (1.): $F\,sy\sqrt{1-s^2 y^2}$ (the whole force):: S. G C $\left(\frac{cy}{\sqrt{1-s^2 y^2}}\right)$: $F\,csy^2$, the force in the direction P Q. And hence to find the mean annual force, we muſt find the ſum of all the $F\,csy^2$ in the circle, or the fluent of $F\,csy^2\,\dot{z} = \frac{F\,csy^2\dot{y}}{\sqrt{1-y^2}}$; whoſe fluent, found as before, is $\frac{1}{2}F\,csz - \frac{1}{2}F\,csy\sqrt{1-y^2}$; and when $y = 1$, the fluent becomes $\frac{1}{4}F\,csp$, and in the whole circle $= F\,csp$; this divided by the whole circumference $2\,p$, the mean force comes out $\frac{1}{2}F\,cs$, that

that is, half the greateſt force, when the ſun is at the greateſt diſtance from the nodes.

Now to compute the force of the ſun to produce the anticipation of the nodes of a ſingle moon at A, the nodes of the orbit being in quadrature; the force of the ſun $= fcs$; the quantity of matter in the moon is $= 1$. Then $g : h :: fcs : \frac{hfcs}{g}$ the ſpace deſcribed in $1''$; and $2p$ (the circumference of a circle whoſe radius is unity, or the diſtance of the moon from the earth) : $360° :: \frac{hfcs}{g} : 360 \times \frac{hfcs}{2pg}$ = the angle deſcribed in $1''$ by the plane of the orbit of a ſolitary moon in ſyzige.

And by a proceſs exactly ſimilar to that uſed before in the caſe of a rigid annulus, it may be ſhewn, that the mean force of the ſun to diſturb the moon, conſtantly in ſyzige, is but half its force when at the greateſt diſtance from the nodes.

It follows therefore, from what has been demonſtrated, that the greateſt force of the ſun to move the annulus in the direction PQA is equal to its greateſt force to move the plane of the moon's orbit, the moon being conſtantly in ſyzige, and that the mean force in both caſes is half the greateſt force; conſequently the mean force of the ſun to move the plane of the annulus in the direction PQA is equal to its mean force to move the plane of a ſolitary moon in ſyzige, in the ſame direction

direction. But by Cor. 2. Prop. 30. Lib. 3. Principia, in any given position of the nodes, the mean horary motion of the nodes of a solitary revolving moon, is just half the horary motion of the nodes of a moon continually in syzige. And Mr. Landen, in his memoirs, has shewn, that when a rigid annulus revolves with two motions, one in its own plane, and the other about one of its diameters, half the whole motive force acting upon the ring is consumed in counteracting the centrifugal force of the ring, by which it endeavours to revolve round a momentary axis, in consequence of its two motions; and the other half only is efficacious in producing the angular motion of the ring about its diameter; so that the motion of the nodes of a detached rigid annulus, being produced by half the mean solar force, is exactly equal to that of the orbit of a solitary moon. For in the case of a solitary moon no centrifugal force to produce a revolution round a momentary axis can take place, there being nothing for the body to act upon; but in a rigid ring, its two motions compounded will give the ring a tendency to revolve about an axis neither perpendicular to nor in the plane of the ring, and therefore this axis cannot be permanent; since each particle of the ring will act by its centrifugal force to impress on it a new motion about an axis perpendicular to the former. But if the rigid annulus, so revolving, be attached to the equator of a sphere, the case will be widely different; for the whole motive force is here employed in giving motion to the annulus and sphere together about

about a diameter of the equator; therefore the part of it which is employed in giving motion to the ring, bears a very ſmall proportion to the whole force, and it is this ſmall part only which is counteracted and rendered inefficient; for the ſphere itſelf has no centrifugal force, whereby it endeavours to revolve round a momentary axis. Hence the motive force being given, viz. the force on the ring, the angular motion generated will be inverſely as the inertia of the matter moved; now the inertia of the annulus is = the matter of the annulus $\times \sqrt{\tfrac{1}{2}}$ (the diſtance of its centre of gyration from the centre of the ring); and the inertia of the ſphere and ring together is = the matter in them $\times \sqrt{\tfrac{2}{5}}$; therefore the angular velocity of the ring muſt be diminiſhed in the ratio of the inertia of the ring to the inertia of the ring and ſphere together, in order to have the angular velocity which now will be produced in the ring, in conſequence of its connection with the ſphere, by the counteracting force. That is, if a be the angular velocity of the ring and ſphere united, the angular velocity which that part of the force which is counteracted could produce in the ring will be $= a \times \frac{\text{inertia of the ring}}{\text{inertia of the ſphere}} = a \times \frac{1}{250}$. The 250th part therefore of the whole force only is now efficient in moving the ring round its diameter; but this part is = the centrifugal force, and therefore it is this part only of the whole ſolar force which is counteracted.

 HENCE

HENCE therefore it appears, that Newton rightly ſuppoſes the preceſſion of the nodes of a rigid, detached annulus, and of a ſolitary moon to be equal; though the principles on which he argues are inſufficient, becauſe he did not, as was neceſſary, conſider the operation of the counteracting centrifugal force. And when he comes to apply this deduction, his concluſion is erroneous, becauſe, omitting the conſideration of the centrifugal force as before, he conceived, that the motion of a ſolitary annulus and of a ring attached to a ſphere were produced by the ſame efficient force; whereas in this latter caſe, the centrifugal force of the annulus vaniſhes, and therefore the whole force of the ſun becomes efficient; that is, the efficient force in the caſe of a ring adhering to the equator of a globe, is double the efficient force in the caſe of a ſolitary ring; and therefore the quantity of the preceſſion, eſtimated on this falſe hypotheſis, comes out too little by juſt one half.

BISHOP HORSELY, in his commentary on this problem, obſerves, that if this aſſertion, to wit, that the motion of the nodes of a rigid annulus and of a ſolitary moon are the ſame, be true, he cannot ſee how the quantity of the preceſſion of the equinoxes can be different from that which is aſſigned by Newton; but he refrains from any abſolute deciſion: "Si hoc "vere dictum ſit (ſays he) ſc. quod par eſt ratio nodorum "annuli lunarum terram ambientis, ſive lunæ illæ ſe mutuo "contingant, ſive liqueſcant, & in annulum continuum for- "mentur,

" mentur, five denique annulus ille rigescat, & inflexibilis
" reddatur, nescio qui fieri possit, ut alius sit punctorum equi-
" noctialium motus a vi solis oriundus, quam calculi Newtoniani
" suadent. Quem tamen longe alium invenere viri permagni
" Eulerus & Simpsonus nostras, quos velim lector consulas.
" Ipse nil definio." Now from what has been said it clearly appears, how the motion of the nodes of a solitary moon and rigid annulus may be equal, and yet the quantity of the precession assigned by Newton erroneous in the ratio of one to two; the efficient motive force of an attached annulus being double the efficient motive force of a ring revolving solitarily, with a compound motion round its centre and one of its diameters.

If then the corrected quantity of 10″ 33‴, be further corrected, by augmenting it in the ratio of two to one, the result will nearly agree with the quantity investigated by other eminent mathematicians; thus Simpson makes it 21″ 7‴, Landen 27″ 7‴, D'Alambert 23″ nearly; Euler 22″; Frisi $21\frac{1}{4}$″; Milner 21″ 6‴, and Mr. Vince, 21″ 6‴; see Phil. Transf. vol. 77.

From this review of the solutions of this problem, it appears that Mr. Landen has the honour of having first detected the particular source of Newton's mistake, by discovering that when a rigid annulus revolves with two motions, one in its own plane and the other round one of its diameters, half the motive force

acting upon the ring is counteracted by the centrifugal force arising from this compound motion, and half only is efficacious in accelerating the plane of the annulus round its diameter. As Mr. Landen has not expressly demonstrated this proposition, I am persuaded I shall afford the mathematical reader much gratification, by here laying before him the following very elegant demonstration, communicated to me by the learned Mr. Brinkley, Professor of Astronomy in the University of Dublin.

PROP. If a rigid ring *n q* N Q revolves with two motions (fig. 3.), one in its own plane, and the other about the diameter *q* T Q; and if a motive force, acting at the point Q, be supposed equivalent to the whole motive force acting upon the ring, then half this force is efficacious in accelerating the motion of the point Q (in a direction perpendicular to the plane of the ring) and the other half is consumed in counteracting the centrifugal force, arising from the motion of the particles of the ring about a momentary axis P T *p*.

IN the great circle *n b* let a point *b* (fig. 3.) be taken indefinitely near to *n*, and in the ring a point *r*, so that *n b* and Q *r* may represent the angular velocities about the diameter and the centre of the ring. Let *d* and *c* represent these velocities, and *r* the radius of the ring. Draw *r s* perpendicular to the plane of the ring, and meeting the great circle *b* Q *s* in *s*; then

then will $r s$ reprefent the accelerating force of the point Q, perpendicular to the plane of the ring; but $rs : nb :: Qr :$ Rad. (r), therefore $r s = \frac{c d}{r}$.

CONSEQUENTLY, if R = the matter of the ring, a motive force acting upon the point Q $= \frac{c d}{r} \times \frac{1}{2}$ R will be equivalent to the whole efficacious motive force on the ring.

THE momentary axis P T p is in a plane perpendicular to the plane of the ring, and which paffes through Qq. Make P T = the radius of the ring, and draw P r perpendicular to Qq, and we have Pr : Tr :: d : c, or P$r = \frac{dr}{\sqrt{c^2 + d^2}}$, and T$r = \frac{c r}{\sqrt{c^2 + d^2}}$. Let P T (in fig. 4.) reprefent the momentary axis, and Q E N a quadrant of the ring. From any point E of the ring draw Ev perpendicular to P T, and $v w$ perpendicular to Q T. The centrifugal force of E : centrifugal force of N :: Ev : N T, or the centrifugal force of E = centrifugal force of N $\times \frac{\mathrm{E}v}{\mathrm{NT}} = \frac{c^2 + d^2}{r} \times$ particle E $\times \frac{\mathrm{E}v}{\mathrm{NT}}$, becaufe the velocity of N $= \sqrt{c^2 + d^2}$. But the efficacious part of this force in a direction perpendicular to the plane of the ring = whole $\times \frac{v w}{\mathrm{E}v}$; and a force acting at Q equivalent

valent to this = whole $\times \frac{vw}{Ev} \times \frac{Tx}{TQ} = \frac{c^2 + d^2}{r} \times E \times \frac{Ev}{NT} \times \frac{vw}{Ev} \times \frac{Tx}{TQ} = \frac{c^2 + d^2}{r} \times E \times \frac{vT \times Pr \times Tx}{TQ^2}$. Now if great circles be conceived drawn through P, Q, and P, E; (by Sph Trig.) cos. PE $(vT) \times$ Rad. (TQ) = cos. PQ $(Tr) \times$ cos. QE (Tx). Therefore a motive force at Q equivalent to the motive, efficient, centrifugal force of E $= \frac{c^2 + d^2}{r} \times E \times \frac{Tr \times Pr \times Tx^2}{TQ^4}$; therefore the ſum of all theſe quantities = the motive force at Q equivalent to the ſum of all the efficient centrifugal forces, or the centrifugal force of the ring. But it is eaſily ſhewn, that the ſum of all theſe quantities $= \frac{c^2 + d^2}{r} \times \frac{1}{2} R \times \frac{Tr \times Pr \times TQ^2}{TQ^4} = \frac{c^2 + d^2}{r} \times \frac{1}{2} R \times \frac{cdr^2 \times TQ^2}{c^2 + d^2 \times TQ^4} = \frac{cd}{r} \times \frac{1}{2}$ R. Hence the motive force at Q, equivalent to the ſum of all the efficacious centrifugal forces, is expreſſed by the ſame quantity $\frac{cd}{r} \times \frac{1}{2}$ R, as the force at Q, equivalent to the whole motive, efficacious force on the ring. Q. E. D.

Mr. Simpson has pointed out the miſtakes in the ſolutions of this problem propoſed by M. Silvabelle and Walmeſley; but neither is his own calculation entirely faultleſs; and his concluſion appears to be correct, only becauſe the errors in the premiſes compenſate each other. Thus he ſuppoſes, that the whole motive force,

force, acting on a detached rigid ring, revolving with a two-fold motion, one round its centre, the other round a diameter, is equal to the efficient force by which the plane of the ring is moved round its diameter; whereas the former is to the latter as two to one; half the whole motive force being counteracted and rendered inefficient by the centrifugal force. 2dly, He ſuppoſes, that the whole efficient motive force, acting on a detached rigid annulus revolving in the ſame manner as before, is equal to the whole efficient motive force acting on an annulus, attached to and connected with a ſphere, which is alſo falſe in the ratio of one to two; the centrifugal force in the caſe of an attached annulus vaniſhing; and therefore no part of the whole force is rendered ineffectual; and conſequently half the motive force in the latter caſe will produce an equal effect as the whole in the former, half of the force in the former caſe not contributing in any degree to the motion of the annulus round its diameter, but being totally employed in counteracting the tendency of the ring to revolve round a momentary axis.

Mr. Milner's and Friſi's calculations become likewiſe correct in the reſult, in the ſame manner as Simpſon's, by the mutual counteraction of equal and contrary errors. Thus they both hold, that the preceſſion of a rigid annulus is double that of a ſolitary moon, whereas they are equal, as we have already demonſtrated, by which the preceſſion would come out twice greater than the truth; but they likewiſe are of opinion, that the pre-

ceſſion

ceſſion of an attached and ſolitary annulus are equal, whereas the former is double that of the latter; this error therefore counterbalances the former.

Mr. Emerson has given two ſolutions of this queſtion, which are both erroneous, one in his Miſcellanies, the other in his Fluxions. In the former he adopts the ſame principles with Newton, in ſuppoſing the preceſſion of a ſolitary moon, a detached rigid annulus, and an attached annulus to be equal. In the latter he determines the direction in which a body would move in conſequence of a uniform motion impreſſed on it in one direction, and a uniformly accelerated motion in another, to be the diagonal of a parallelogram, whoſe two ſides repreſent the ſpaces deſcribed from quieſcence, in the ſame time, by the two forces; which, as Mr. Milner has juſtly obſerved, produces an error of one half in the concluſion. For let A D be the ſpace deſcribed by the uniform motion (fig. 5.), while the body would deſcribe A B by the accelerated motion; ſince the time is indefinitely little, the accelerating force may be conſidered as conſtant, and therefore the body will in fact deſcribe the parabola A G C; and the direction of the motion at C will be the tangent E C; but the angle $DEC = DAC + ACE = 2\,DAC$ nearly, becauſe the tangents A E, C E, are very nearly equal (Ham. Con. Cor. 1. Prop. 3. Lib. 2. and Prop. 3. Lib. 3.); that is, the true angle of deviation D E C, is very nearly double the angle of

of deviation DAC, as determined by the diagonal of the parallelogram.

In this solution Mr. Emerſon ſays, "the earth being an oblate "ſpheroid, the ſphere is encompaſſed with a ſolid cruſt going "round the equator in the manner of a ring; now the effect of "the forces of the ſun and moon upon this cruſt, and the motion "communicated thereby to the whole body of the earth, is what "we are to enquire after." He then calculates the force of the ſun upon the annulus, and ſuppoſes this whole force efficient; he next ſuppoſes this whole motive force to act at the diſtance of the centre of gyration from the centre of the earth, and thence deduces the motion generated in the plane of the equator about one of its diameters. It appears therefore, that he ſuppoſes the whole motive force of the ſun to be efficient on the annulus, ſeparately conſidered: and 2dly, that this efficient force is equal to the efficient force on the ſame annulus, when connected with the earth; which, excluſive of the error detected by Mr. Milner, are the very ſame falſe hypotheſes with thoſe adopted by Simpſon.

But here a queſtion naturally ariſes, if the error of Newton's calculation be as great as is pretended, whence comes it to paſs that the reſult of his calculation agrees ſo exactly with phænomena; for on ſuppoſition, that the preceſſion ariſing from the force of the ſun alone is but $9'' \, 7'''$, the preceſſion cauſed by

the moon will be $40'' \, 52''' \, 52^{iv}$, and the whole preceſſion, ariſing from both cauſes conjoined, will be $50'' \, 0''' \, 12^{iv}$, according to obſervation.

To this objection a ſatisfactory anſwer is ſuggeſted by Newton himſelf, where he ſays, that the preceſſion will be diminiſhed if the matter of the earth be rarer at the circumference than at the centre. The reaſon of which is evident from what has been already demonſtrated, for the quantity of matter in the earth being given, the diſtance of the centre of gyration from the centre of the earth will be leſs, the more the matter of the earth is accumulated towards the centre, and therefore the leſs will be the angular motion generated by the ſun and moon.

Fig. 1.

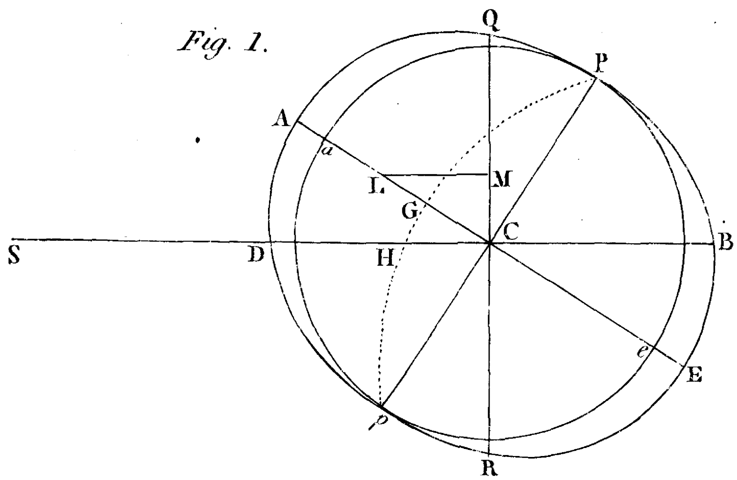
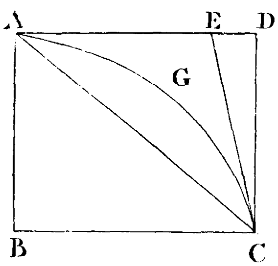


Fig. 5.



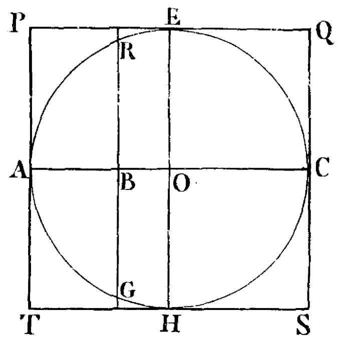
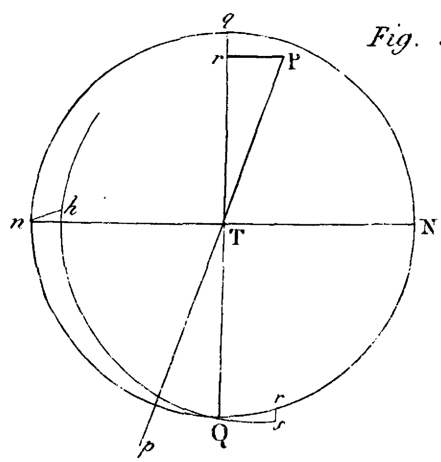


Fig. 2.

Fig. 3.



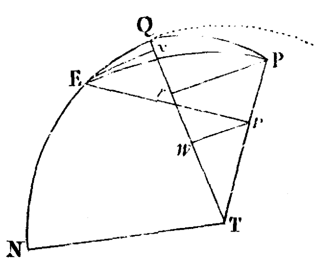


Fig. 4.